Anomalous Thresholds and Three-Particle Unitarity Integral*

Y. S. Kim
Department of Physics and Astronomy, University of Maryland, College Park, Maryland
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Anomalous thresholds of the square diagram and their effect on the two-particle unitarity integral are discussed. It is shown for a certain diagram that the three-particle unitarity integral does not have counter terms which cancel the singularity from the two-particle contribution. Consistency with the crossed-channel unitarity is also discussed.

I. INTRODUCTION

ONE of the outstanding difficulties in dispersion theory is the problem of the unitarity integral involving more than two particles in the intermediate state. The difficulty, which arises from both kinematical complications and analytic properties of the production amplitude,[1] has kept us from going any further than a two-particle approximation of the many particle system.[2]

For the three-particle intermediate state, much work has been done by Gribov *et al.* for simple Feynman diagrams.[3] In an attempt to obtain the Mandelstam double spectral function, they investigate analytic properties of the three-particle unitarity integral.

In this note, we use the method of Gribov *et al.* to study the effect of the leading anomalous curve of the four-point function on the two-particle unitarity integral. Using a kinematically simple diagram as an illustration, we show that the three-particle unitarity integral has no terms which would cancel the singularity coming from the two-body integral.

II. STATEMENT OF THE PROBLEM

We consider the absorptive part of the amplitude associated with the sixth-order diagram of Fig. 1. As usual, the amplitude is regarded as a function of the two invariants

$$s=(p_1+p_2)^2=(p_3+p_4)^2,\qquad t=(p_2-p_4)^2=(p_1-p_3)^2. \tag{1}$$

For sufficiently large values of s, one should take into account both two- and three-particle intermediate states in the s-channel unitarity integral. We discuss, in particular, the effect of anomalous thresholds of the fourth-order diagram and their counter part coming from the three-particle intermediate state.

For simplicity, we assume that all particles, except that of line 9, have unit mass. The fourth-order diagram,

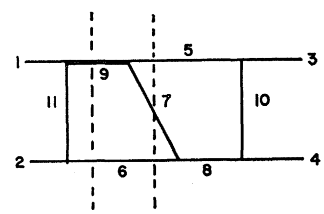

FIG. 1. Feynman diagram under consideration. The dashed lines indicate the unitarity cuts.

resulting from the unitarity cut across lines 6 and 9, develops anomalous thresholds in both the s and $(p_4-p_6)^2$ variables as the mass variable of the "external line" 9, $M_9{}^2$, becomes greater than 3. We denote this mass variable by ω. The anomalous thresholds are real if ω is smaller than 4. In the present discussion we restrict ourselves to the real anomalous threshold, i.e.,

$$3<\omega<4. \tag{2}$$

Regarded as a function of $\tau=(p_4-p_6)^2$, the amplitude for the box diagram has Landau singularities at

$$\begin{aligned}\tau&=T_4=4,\\ \tau&=T_3=1+\tfrac{1}{2}\omega+\tfrac{1}{2}[3\omega(4-\omega)]^{1/2},\\ \tau&=T_4\equiv\frac{s(2s-5-\omega)+[s^2(2s-5-\omega)^2+3s(s-4)(\omega-1)^2]^{1/2}}{s(s-4)},\end{aligned} \tag{3}$$

corresponding, respectively, to the normal threshold, the triangle anomalous threshold, and the singularity of the leading Landau curve.[4] After performing the two-particle unitarity integral one can show that regarded as a function of t, the absorptive part corresponding to the whole diagram has singularities at

$$t_i=\frac{s(s-\omega-1)(T_i-1)+2[(s-\omega-1)^2+\omega(s-4)]}{(s-\omega-1)^2-4\omega}+\frac{2\{[(s-\omega-1)^2+\omega(s-4)][s(T_i-1)^2+s(s-\omega-1)(T_i-1)+(s-\omega-1)^2+\omega(s-1)]\}^{1/2}}{(s-1-\omega)^2-4\omega}, \tag{4}$$

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[1] L. F. Cook and Jan Taroki, J. Math. Phys. **3**, 1 (1962); Y. S. Kim, Phys. Rev. **124**, 1632 (1961).
[2] L. F. Cook and B. W. Lee, Phys. Rev. **127**, 283 (1962).
[3] V. N. Gribov and I. T. Dyatlov, Zh. Eksperim. i Teor. Fiz. **42**, 196 (1962); **42**, 1268 (1962) [translation: Soviet Phys.—JETP, **15**, 140 (1962); **15**, 879 (1962)].
[4] J. Tarski, J. Math. Phys. **1**, 149 (1960).

unless other unitarity integrals have counter terms. Here i takes the values 2, 3, or 4.

As s goes to infinity, t_2 and t_4 approach 9, which is the normal threshold for the crossed t channel. In the case of the triangle singularity, t_3 gives rise to a singularity curve in the st plane which comes down asymptotically below $t=9$. Gribov *et al.*, however, showed that this triangle singularity is cancelled by the counter term coming from the three-particle unitarity integral.[3]

In this note, we extend the double-pinch argument of Gribov to the determination of the singularity nature of the contribution from the leading Landau curve of the box diagram. It will be shown that, contrary to the previous case, the three-particle unitarity integral has *no* singularities which would cancel t_4, and, therefore, that the leading curve t_4 is the singularity curve of the *whole* diagram.

We observe here that T_4 is smaller than 4 for the physical values of s under consideration and that it approaches 4 as s goes to infinity. It will also be shown that, in spite of this, the leading curve never comes down below $t=9$. This conclusion is consistent with the unitarity in the crossed t channel.

III. SOLUTION TO THE PROBLEM

Let us now consider the unitarity cut across the three lines 5, 7, and 6. According to Gribov *et al.*, the unitarity integral takes the form

$$A^{(3)}(s,t)=\int_4^{(\sqrt{s}-1)^2}\frac{d\beta}{\beta-\omega}\int_{\gamma-}^{\gamma+}\frac{d\gamma}{[K_1(t,\beta,\gamma)]^{1/2}} \times\int_{\alpha-}^{\alpha+}\frac{d\alpha}{(\alpha-1)[K_2(\gamma,\alpha,\beta)]^{1/2}}, \quad (5)$$

where

$$\alpha_\pm=\tfrac{1}{2}(3+s-\beta) \pm\tfrac{1}{2}\{(\beta-4/\beta)[(s+1-\beta)^2-4s]\}^{1/2},$$

$$\gamma_\pm=\tfrac{1}{2}(3+\beta-s) \pm\tfrac{1}{2}\{(s-4/s)[(s+1-\beta)^2-4s]\}^{1/2},$$

$$K_1(t,\beta,\gamma)=4s(s-4)\gamma^2+4s[2(s-4)+t(s-1-\beta)]\gamma -[(s+1-\beta)^2-4s]t^2+4[2(s+1-\beta)^2 -8s-s(s-1-3\beta)]t-4s(s-4),$$

$$K_2(\gamma,\alpha,\beta)=[\gamma^2-2\gamma(\beta+1)t(\beta-1)^2]\alpha^2 -2\{\gamma s(\gamma-\beta-1)+[2(\beta-1)^2 +\gamma[\gamma-3-\beta)]\}\alpha+(s-1)^2(\gamma-4)\gamma.$$

Constant factors have been suppressed in Eq. (5).

Evaluating the last integral, we obtain the double integral form

$$A^{(3)}(s,t)=\int_4^{(\sqrt{s}-1)2} d\beta\,\frac{F(\beta,s,t)}{\beta-\omega}, \quad (6)$$

where

$$F(\beta,s,t)=\int_{\gamma-}^{\gamma+}\frac{d\gamma}{[K_1(t,\beta,\gamma)H(\beta,\gamma)]^{1/2}}\ln\frac{(\alpha_--1)\{f^{(+)}(\beta,\gamma)+[H(\beta,\gamma)]^{1/2}g^{(+)}(\beta,\gamma)\}}{(\alpha_+-1)\{f^{(-)}(\beta,\gamma)+[H(\beta,\gamma)]^{1/2}g^{(-)}(\beta,\gamma)\}},$$

$$f^{(\pm)}(\beta,\gamma)=s(\alpha_\pm+3-s)\gamma^2+(\beta-1)^2(\alpha_\pm+2)+\{\alpha_\pm[(\beta-1)-s(\beta+1)]+[4(s-1)^2-s(\beta+1)-(\beta-3)]\}\gamma, \quad (7)$$

$$g^{(\pm)}(\beta,\gamma)=\tfrac{1}{2}\{(s-1-\alpha_\pm)[(s+1-\beta)^2-4s]^{1/2}-\tfrac{1}{2}(2\gamma-3+s-\beta)[[(s+1-\beta)^2-4s]^{1/2}\mp(s+\beta-1)(\beta-4/\beta)^{1/2}]\},$$

$$H(\beta,\gamma)=s(s-4)\gamma^2-2s(2s-5-\beta)\gamma-3(\beta-1)^2.$$

Again, constant factors have been suppressed in Eq. (6).

In the following, we shall first study the integrand in Eq. (7). We shall then investigate analytic properties of $F(\beta,s,t)$ and, eventually, the absorptive part $A^{(3)}(s,t)$.

Let us first consider the singularity curves of the integrand in the $\gamma\beta$ plane. The logarithmic function gives logarithmic singularities on the ellipse

$$(\gamma-2)^2+(\beta-2)^2-(\gamma-2)(\beta-2)=3. \quad (8)$$

We shall call this "ellipse $\mathcal{E}$."

The $H(\beta,\gamma)$ factor inside the square root sign of the denominator together with the logarithmic function gives singularities on the section of the hyperbola

$$s(s-4)\gamma^2-2s(2s-5-\beta)\gamma-3(s-1)^2=0, \quad (9)$$

which is tangent to ellipse $\mathcal{E}$ at

$$\beta=\beta_t\equiv[3(s-1)^2/s^2-3s+3]. \quad (10)$$

We call this "hyperbola $h(s)$." As we increase s, this tangent point moves down and reaches its asymptotic value 3 for large s.

Now for a fixed β, $0\leqq\beta\leqq(\sqrt{s}-1)^2$, the singularity

$$\gamma(t)=1+\frac{t(s-\beta-1)}{s-4} -\frac{2}{s-4}\{t(t+s-4)[(s+1-\beta)^2+s(\beta-4)]\}^{1/2}, \quad (11)$$

due to the $K_1(t,\beta,\gamma)$ factor, moves from the right toward the upper limit of the γ integration as we increase t along the real axis. This singularity then reaches the upper end and drags the contour of the γ integration to the right. As $\gamma(t)$ reaches a singularity on either ellipse $\mathcal{E}$ or hyperbola $h(s)$, the γ integration is pinched. In the $\gamma\beta$ plane of Fig. 2, singularities due to this coincidence are given by the ordinates of the points at which the curve $\gamma(t)$ intersects ellipse $\mathcal{E}$ or hyperbola $h(s)$. The $\gamma(t)$ hyperbola intersects the ellipse twice. We call the ordinates of the intersection points $\beta_1^{(+)}(s,t)$ and $\beta_1^{(-)}(s,t)$ with $\beta_1^{(+)}(s,t)>\beta_{1,}^{-)}(s,t)$. The $\gamma(t)$ curve intersects hyperbola $h(s)$ at $\beta_2^{(+)}(s,t)$ and $\beta_2^{(-)}(s,t)$. Here again $\beta_2^{(+)}(s,t)>\beta_2^{(-)}(s,t)$.

For the interval $\gamma_+ \leqq \gamma < \gamma(t)$, we take the discontinuity of the integrand due to the branch point $\gamma(t)$ and write the whole γ integral from γ_- to $\gamma(t)$.

Let us now turn to the β integration and analytic properties of $A^{(3)}(s,t)$. As we increase t, the singular point $\beta_1^{(+)}(s,t)$ moves up along the left branch of ellipse $\mathcal{E}$, reaches the lower limit of the β integration, $\beta=4$, and drags down the β contour as it moves down along the right branch of ellipse $\mathcal{E}$. $\beta_1^{(-)}(s,t)$ moves up along the right branch in the meantime. When $\beta_1^{(+)}$ coincides with $\beta_1^{(-)}$, the coincidence gives the singularity corresponding to the leading Landau curve of the diagram in which lines 8 and 9 are reduced. Let us call this coincidence point $\beta(s)$. The $\gamma(t)$ curve is tangent to ellipse $\mathcal{E}$ at $\beta(s)$. It is easy to show that $\beta(s)$ is a monotonically decreasing function of s and reaches 3 as s goes to infinity.[5] As one increases t further, both $\beta_1^{(+)}$ and $\beta_1^{(-)}$ go over to the complex plane.

Now if s is such that $\beta(s)<\omega$, $\beta_1^{(+)}$ reaches ω before going over to the complex plane, then the coincidence gives rise to the singularity which cancels its counter part on the curve t_3 coming from the two-particle unitarity integral.[3]

However, if s is such that

$$\beta_t > \beta(s)\,, \quad \beta_t > \omega\,, \tag{12}$$

$\beta_2^{(+)}$ may catch the β integration when $\beta_1^{(+)}$ reaches the tangent point β_+ and then drag the contour along hyperbola $h(s)$. (See Fig. 3.) As we increase t further $\beta_2^{(+)}$ will move down and coincide with the pole at $\beta=\omega$ to give another singularity in the t plane. This singularity will then be the counter term to the one on the curve mentioned in Sec. II. We stress here that the inequalities in Eq. (12) are the necessary conditions for $A^{(3)}$ to have the counter term in question.

Using both numerical method and the asymptotic

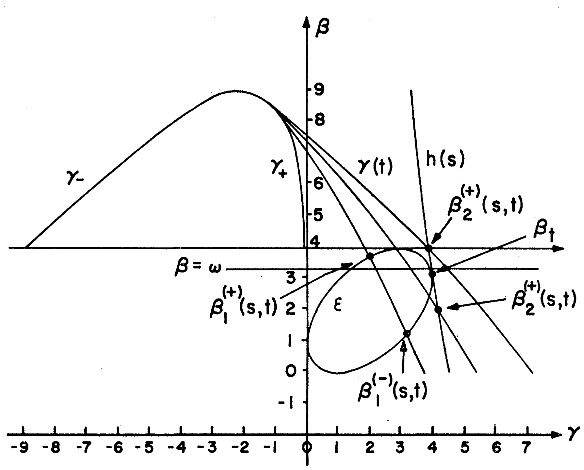

FIG. 2. Singularity curves in the $\gamma\beta$ plane. All points of intersection are denoted by their ordinates. The drawing was made at $s=16$.

[5] J. C. Polkinghorne, *Lectures in Theoretical Physics* (W. A. Benjamin, Inc., New York, 1962), Vol. 1, p. 152.

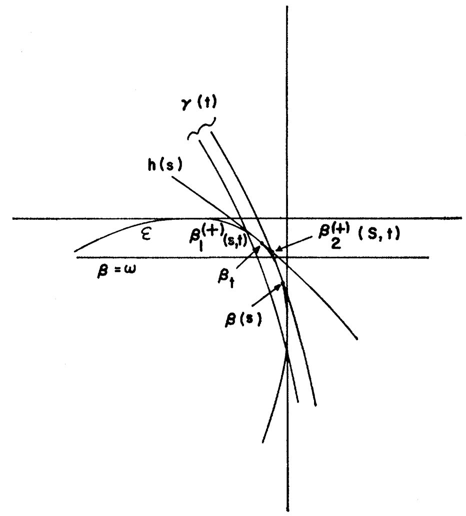

FIG. 3. The case in which $\beta_2^{(+)}(s,t)$ may drag the integration contour along hyperbola $h(s)$.

expression for β_t and $\beta(s)$ for large s:

$$\beta_t \simeq 3(1+1/s)\,, \quad \beta(s) \simeq 3(1+25/s)\,,$$

we can show that the inequality

$$\beta_t > \beta(s)$$

is not satisfied for the physical values of s under consideration. Therefore, there are no counter terms in the three-particle unitarity integral which would cancel the t_4 singularity from the two-particle intermediate state.

We point out further that the t_4 curve lies entirely above $t=9$ with the asymptote for large s

$$t \simeq 9 + (1/s)(7\omega+18)\,.$$

This can also be shown by numerical work.

A similar analysis may be made for the two-body unitarity cut across the internal lines 5 and 8, and also for the three-body cut across 9, 7, and 8. Singularity obtained in this analysis, if not cancelled, should also be included as the singularities of the whole amplitude. This, however, does not affect the above conclusion.

The t_4 singularity is, thus, the singularity of the whole amplitude and lies entirely above $t=9$ in the st plane. The present result supports the argument that the anomalous threshold of the square part does not give any anomalies for the whole diagram.

The preceding discussion hinges upon the fact that hyperbola $h(s)$ is tangent to ellipse $\mathcal{E}$ and also that one can locate the tangent point by solving a fourth-degree equation. For the general mass case, however, it seems difficult to tell whether curves $\mathcal{E}$ and $h(s)$ are tangent or whether, even if they are, the tangent point can be located by an elementary method.

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